(2.2), (3.3) and (5.1) show that the functional E_λ is also positive-definite for all possible Lagrangian displacement fields $\xi(x)$ and velocity fields $\mathbf{u}(x)$.

Consequently, the estimate

$$E_{\Lambda^++\varepsilon}(t) \leqslant \mathring{E}_{\Lambda^++\varepsilon} \exp[2(\Lambda^+ + \varepsilon)t]$$

follows from (3.4) when $\lambda = \Lambda^+ + \varepsilon$ $(\varepsilon > 0)$ which, by using the inequality $\Pi_{\Lambda^+}(t) \geqslant 0$, is transformed to the more obvious form

$$2K_{\Lambda^++\varepsilon}(t) + \varepsilon(2\Lambda^+ + \varepsilon)M(t) + \varepsilon G \leqslant 2\mathring{E}_{\Lambda^++\varepsilon} \exp[2(\Lambda^+ + \varepsilon)t] \tag{5.2}$$

It is seen from (5.2) that the parameter $\Lambda^+ + \varepsilon$ gives an upper estimate of the increments in the solutions of problem (2.1). A comparison of estimates (4.11) and (5.2), taking account of (4.12), shows that the parameter Λ^+ gives both an upper and lower estimate of the rate of growth in the most critical perturbations (4.8):

$$\Lambda^+ - \delta \leqslant \omega_* \leqslant \Lambda^+ + \varepsilon$$

The author thanks V.A. Vladimirov for suggesting the problem and for useful discussions.

REFERENCES

1. CHATAYEV N.G., The Stability of Motion, Nauka, Moscow, 1965.
2. MOISEYEV N.N. and RUMYANTSEV V.V., Dynamics of a Body with Cavities containing a Fluid, Nauka, Moscow, 1965.
3. VLADIMIROV V.A. and RUMYANTSEV V.V., On the converse Lagrange theorem for a solid with a hollow containing an ideal fluid, Prikl. Matem. i Mekhan., 53, 4, 1989.
4. VLADIMIROV V.A. and RUMYANTSEV V.V., On the converse Lagrange theorem for a solid with a hollow containing a viscous fluid, Prikl. Matem. i Mekhan., 54, 2, 1990.
5. LANDAU L.D. and LIFSHITZ E.M., Theoretical Physics. 8, Electrodynamics of Continuous Media, Nauka, Moscow, 1982.
6. DEMUTSKII V.P. and POLOVIN R.V., Fundamentals of Magnetohydrodynamics, Energoatomizdat, Moscow, 1987.

Translated by E.L.S.

PMM U.S.S.R.,Vol.54,No.6,pp.815-820,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00
©1992 Pergamon Press plc

THE CONSTRUCTION OF RECIPROCITY AND INTEGRAL REPRESENTATION FORMULAS OF THE GENERAL SOLUTION FOR QUASISTATIC AND DYNAMIC PROBLEMS OF UNCOUPLED GENERALIZED THERMOELASTICITY*

YU.M. MAMEDOV

Reciprocity formulas are constructed and representations of the Somigliani-type are obtained for quasistatic and dynamic problems of uncoupled generalized thermoelasticity in the Lord-Shulman formulation that is effective for applications. Moreover, representations are obtained for the stresses and heat flux. Unlike the existing approach (/1/, say) these formulas are derived on the basis of an examination of the system of differential equations of the above-mentioned problems of generalized thermoelasticity as a system with appropriate non-selfadjoint differential operators. Operators adjoint to the initial differential operators are introduced into consideration for the construction of the reciprocity formulas (second Green's formula), and a Laplace transformation is used.

**Prikl.Matem.Mekhan.*,54,6,992-997,1990

1. *Formulation of the problem.* The system of equations for dynamic problems uncoupled generalized thermoelasticity (UGT) has the form

$$\rho u_i^{\cdot\cdot} - (C_{ijkl} u_{k,l})_{,j} + (\beta_{ij} T)_{,j} = X_i \tag{1.1}$$
$$c_\varepsilon \tau_t T^{\cdot\cdot} + c_\varepsilon T^{\cdot} - (\lambda_{kj} T_{,j})_{,k} = G \ (i, j, k, l = 1, 2, 3)$$

Here u_i and X_i are the components of the displacement vector and mass force vector, G is the volume density of heat sources, T is the deviation of the actual absolute temperature from the absolute temperature Θ_0 of the undeformed state, C_{ijkl}, λ_{kj}, β_{ij}, c_ε and mechanical and thermophysical characteristics of the body, τ_t is a relaxation constant and ρ is the density of the medium.

The first equation of (1.1) is obtained from the equations of motion $(\sigma_{ij,j} + X_i = \rho u_i^{\cdot\cdot})$ as a result of substituting the Duhamel-Neumann expression for the stresses

$$\sigma_{ij} = {}^1/_2 C_{ijkl} (u_{k,l} + u_{l,k}) - \beta_{ij} T \tag{1.2}$$

where $p_i = \sigma_{ij} n_j$.

We consider the system of Eqs.(1.1) in the whole space R^3 as well as in the domain $V \subset R^3$. It is assumed that the domain possesses a piecewise-smooth boundary S. The points of space are denoted by $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ while $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal to the boundary S (exterior with respect to the body V). It is considered that the tensors C_{ijkl}, β_{ij}, λ_{kj} are symmetric.

Let the boundary of the domain V be separated into four parts

$$S = S^1 \cup S^2 \cup S^3 \cup S^4, \quad S^i \cap S^j = \varnothing \ (i, j = 1, 2, 3, 4;\ i \neq j)$$

where displacements and temperatures are given on S^1, surface forces and heat flux on S^2, displacements and heat flux on S^3, and surface forces and temperatures on S^4:

$$\mathbf{u} = \mathbf{f},\ T = g \quad \text{on} \quad S^1;\ \mathbf{p} = \mathbf{h},\ Q = d \quad \text{on} \quad S_R^2 \tag{1.3}$$
$$\mathbf{u} = \mathbf{f},\ Q = d \quad \text{on} \quad S_R^3;\ \mathbf{p} = \mathbf{h},\ T = g \quad \text{on} \quad S_R^4$$

Moreover

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ(\mathbf{x}),\ \mathbf{u}^{\cdot}(\mathbf{x}, 0) = \mathbf{u}^1(\mathbf{x}) \tag{1.4}$$
$$T(\mathbf{x}, 0) = T^\circ(\mathbf{x}),\ T^{\cdot}(\mathbf{x}, 0) = T^1(\mathbf{x}),\ \mathbf{x} \in V$$

where

$$\mathbf{p} = T_{\mathbf{n}}\mathbf{u} - \beta \mathbf{n} T, \quad Q = \partial T / \partial n^+$$
$$\frac{\partial}{\partial n^+} = \sum_{k, j=1}^{3} \lambda_{kj}(\mathbf{x})\, n_k(\mathbf{x}) \frac{\partial}{\partial x_j}$$

Here $\mathbf{T_n}$ is the stress operator, S_R^2, S_R^3 and S_R^4 is the set of all points on S^2, S^3, S^4, respectively, at which the normal is defined, and g, d, $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{h} = (h_1, h_2, h_3)$ are scalar and vector functions given on the boundary.

It is assumed that the functions **u**, **p**, T and Q on the surface S are sufficiently smooth so that integrals of the potential type, in which these functions are taken as densities exist.

The initial-boundary value problems for this formulation are as follows. Find the thermoelastic state $(\mathbf{u}, \sigma, T)$ of a medium in a time interval $[t_0, t_1]$, corresponding to a mass force X, a thermal source G, and the initial conditions (1.4) (the first two conditions in (1.4) are missing in the case of quasistatic problems) and according to the appropriate boundary conditions (1.3) for $S^m = S$, where m is the number of the problem.

We will say that a mixed initial-boundary value problem of uncoupled thermoelastic dynamics (or quasistatics) holds if at least two of the four parts of the boundary S are considered.

2. *Reciprocity formulas.* To construct the reciprocity formulas we write the system of equations adjoint to (1.1)

$$\rho u_i^{\prime\cdot\cdot} - (C_{ijkl} u_{k,l}')_{,j} = X_i' \tag{2.1}$$
$$c_\varepsilon \tau_t T^{\prime\cdot\cdot} + c_\varepsilon T^{\prime\cdot} - (\lambda_{kj} T_{,j}')_{,k} - {}^1/_2 \beta_{kj} (u_{k,j}' + u_{j,k}') = G$$

For the adjoint system

$$\sigma_{ij}' = {}^1/_2 C_{ijkl} (u_{k,l}' + u_{l,k}'), \quad p_i' = \sigma_{ij}' n_j \tag{2.2}$$

Applying a Laplace transformation to (1.2) and the first equation in (2.2), carrying out all the necessary calculations, using Green's formula and the equilibrium equation, we obtain the first part of the reciprocity formula (everywhere henceforth the volume integrals are evaluated over the volume V and the surface integrals over the surface S)

$$\int (\bar{u}_i'\bar{X}_i - \bar{u}_i\bar{X}_i')\,dV + \int (\bar{u}_i'\bar{p}_i - \bar{u}_i\bar{p}_i')\,dS + \int \rho\,(p^*\bar{u}_i'u_i^\circ + \bar{u}_i'u_i^1)\,dV = -\int \beta_{ij}\bar{\varepsilon}_{ij}'\bar{T}\,dV \tag{2.3}$$

We obtain the second part of the reciprocity formula from the heat conduction equations

$$\int (\bar{T}\bar{Q}' - \bar{T}'\bar{Q})\,dS - \int (\bar{G}\bar{T}' - \bar{G}'\bar{T})\,dV - \int c_\varepsilon T^\circ \bar{T}'\,dV - \int \tau_t c_\varepsilon p^* \bar{T}' T^\circ\,dV - \int \tau_t c_\varepsilon T^1 \bar{T}'\,dV = -\int \beta_{kj}\bar{\varepsilon}_{kj}'\bar{T}\,dV \tag{2.4}$$

Eliminating the integral on the right-hand side from (2.3) and (2.4), applying an inverse Laplace transformation to the equality obtained, we find the reciprocity formula for dynamic UGT problems

$$\begin{aligned}
&\int_0^t\!\!\int [u_i(\mathbf{x}, t-\tau)\,X_i'(\mathbf{x},\tau) - X_i(\mathbf{x},\tau)\,u_i'(\mathbf{x}, t-\tau)]\,dV_\mathbf{x}\,d\tau + \\
&\int_0^t\!\!\int [G'(\mathbf{x}, t-\tau)\,T(\mathbf{x},\tau) - G(\mathbf{x},\tau)\,T'(\mathbf{x}, t-\tau)]\,dV_\mathbf{x}\,d\tau - \\
&\int \rho\,[u_i^{\cdot\prime}(\mathbf{x}, t)\,u_i^\circ(\mathbf{x}) + u_i'(\mathbf{x}, t)\,u_i^1(\mathbf{x})]\,dV_\mathbf{x} = \\
&\int_0^t\!\!\int [T'(\mathbf{x}, t-\tau)\,Q(\mathbf{x},\tau) - Q'(\mathbf{x}, t-\tau)\,T(\mathbf{x},\tau)]\,dS_\mathbf{x}\,d\tau + \\
&\int \tau_t c_\varepsilon(\mathbf{x})\,T^1(\mathbf{x})\,T'(\mathbf{x}, t)\,dV_\mathbf{x} + \int \tau_t c_\varepsilon(\mathbf{x})\,T^\circ(\mathbf{x})\,T^{\cdot\prime}(\mathbf{x}, t)\,dV_\mathbf{x} + \\
&\int c_\varepsilon(\mathbf{x})\,T^\circ(\mathbf{x})\,T'(\mathbf{x}, t)\,dV_\mathbf{x} + \int_0^t\!\!\int [u_i'(\mathbf{x}, t-\tau)\,p_i(\mathbf{x},\tau) - \\
&p_i'(\mathbf{x}, t-\tau)\,u_i(\mathbf{x},\tau)]\,dS_\mathbf{x}\,d\tau
\end{aligned} \tag{2.5}$$

In the case of quasistatic UGT problems (here $\rho u_i^{\cdot\cdot} \equiv 0$) in the first equation in (1.1)), the reciprocity formula has the following form (it is constructed in exactly the same way as for the preceding case)

$$\begin{aligned}
&\int [u_i(\mathbf{x}, t)\,X_i'(\mathbf{x}) - X_i(\mathbf{x}, t)\,u_i'(\mathbf{x})]\,dV_\mathbf{x} + \\
&\int_0^t\!\!\int [G'(\mathbf{x}, t-\tau)\,T(\mathbf{x},\tau) - G(\mathbf{x},\tau)\,T'(\mathbf{x}, t-\tau)]\,dV_\mathbf{x}\,d\tau = \\
&\int_0^t\!\!\int [T'(\mathbf{x}, t-\tau)\,Q(\mathbf{x},\tau) - Q'(\mathbf{x}, t-\tau)\,T(\mathbf{x},\tau)]\,dS_\mathbf{x}\,d\tau + \\
&\int c_\varepsilon(\mathbf{x})\,T^\circ(\mathbf{x})\,T'(\mathbf{x}, t)\,dV_\mathbf{x} + \int \tau_t c_\varepsilon(\mathbf{x})\,T^1(\mathbf{x})\,T'(\mathbf{x}, t)\,dV_\mathbf{x} + \\
&\int \tau_t c_\varepsilon(\mathbf{x})\,T^\circ(\mathbf{x})\,T^{\cdot\prime}(\mathbf{x}, t)\,dV_\mathbf{x} + \int [u_i'(\mathbf{x})\,p_i(\mathbf{x}, t) - p_i'(\mathbf{x})\,u_i(\mathbf{x}, t)]\,dS_\mathbf{x}
\end{aligned} \tag{2.6}$$

We note that for $\tau_t = 0$ formulas (2.5) and (2.6) agree with the reciprocity formulas of the quasistatic and dynamic problems of classical uncoupled thermoelasticity /5/*. (*See also: Mamedov. Yu.M. Application of the potential method in thermoelasticity problems. Preprint No.236, Inst. Fiz., Akad. Nauk Azerb.SSR, Baku, 1987).

3. ***Integral representations of the general solution.*** To construct formulas of the Somigliani-type the reciprocity Eqs.(2.5) and (2.6) are used.

We replace x and y in (2.5) and set therein

$$\begin{aligned}
X_i' &= \delta_{im}\delta(\mathbf{y}-\mathbf{x})\,\delta(t-\tau), \quad G' = 0 \\
u_i' &= U_{im}(\mathbf{y}, \mathbf{x}, t-\tau), \quad p_i' = T_{im}(\mathbf{y}, \mathbf{x}, t-\tau) \\
T' &= K_m(\mathbf{y}, \mathbf{x}, t-\tau), \quad Q' = N_m(\mathbf{y}, \mathbf{x}, t-\tau)
\end{aligned}$$

We then obtain the following integral formula for the dynamic UGT problems (in the displacements)

$$u_m(\mathbf{x}, t) = \int_0^t\!\!\int U_{im}(\mathbf{y}, \mathbf{x}, t-\tau)\,X_i(\mathbf{y},\tau)\,dV_\mathbf{y}\,d\tau + \int_0^t\!\!\int G(\mathbf{y},\tau)\,K_m(\mathbf{y}, \mathbf{x}, t-\tau)\,dV_\mathbf{y}\,d\tau + \tag{3.1}$$

$$\int \{\rho(\mathbf{y})[\dot{U}_{im}(\mathbf{y},\mathbf{x},t)u_i^\circ(\mathbf{y}) + U_{im}(\mathbf{y},\mathbf{x},t)u_i^1(\mathbf{y})] +$$
$$c_\varepsilon(\mathbf{y})[K_m(\mathbf{y},\mathbf{x},t)T^\circ(\mathbf{y}) + \tau_t T^1(\mathbf{y})K_m(\mathbf{y},\mathbf{x},t) + \tau_t T^\circ(\mathbf{y})K_m^{\cdot}(\mathbf{y},\mathbf{x},t)]\}\,dV_{\mathbf{y}} +$$
$$\int_0^t\int [K_m(\mathbf{y},\mathbf{x},t-\tau)Q(\mathbf{y},\tau) - N_m(\mathbf{y},\mathbf{x},t-\tau)T(\mathbf{y},\tau)]\,dS_{\mathbf{y}}\,d\tau +$$
$$\int_0^t\int [U_{im}(\mathbf{y},\mathbf{x},t-\tau)p_i(\mathbf{y},\tau) - T_{im}(\mathbf{y},\mathbf{x},t-\tau)u_i(\mathbf{y},\tau)]\,dS_{\mathbf{y}}\,d\tau$$
$$N_m(\mathbf{y},\mathbf{x},t) = \partial K_m(\mathbf{y},\mathbf{x},t)/\partial n^+(\mathbf{y}),\ \mathbf{x}\in V,\ t>0$$

In the case when

$$X_i' = 0,\quad G' = \delta(\mathbf{y}-\mathbf{x})\,\delta(t-\tau)$$
$$u_i' = p_i' = 0,\ T' = T^*(\mathbf{y},\mathbf{x},t-\tau);\ Q' = Q^*(\mathbf{y},\mathbf{x},t-\tau)$$

we obtain Somigliani-type formulas for the temperature from the reciprocity Eq.(2.5)

$$T(\mathbf{x},\mathbf{y}) = \int_0^t\int T^*(\mathbf{y},\mathbf{x},t-\tau)G(\mathbf{y},\tau)\,dV_{\mathbf{y}}\,d\tau + \tag{3.2}$$
$$\int c_\varepsilon(\mathbf{y})T^*(\mathbf{y},\mathbf{x},t)T^\circ(\mathbf{y})\,dV_{\mathbf{y}} + \int \tau_t c_\varepsilon(\mathbf{y})T^1(\mathbf{y})T^*(\mathbf{y},\mathbf{x},t)\,dV_{\mathbf{y}} +$$
$$\int \tau_t c_\varepsilon(\mathbf{y})T^\circ(\mathbf{y})\dot{T}^*(\mathbf{y},\mathbf{x},t)\,dV_{\mathbf{y}} + \int_0^t\int [T^*(\mathbf{y},\mathbf{x},t-\tau)Q(\mathbf{y},\tau) -$$
$$Q^*(\mathbf{y},\mathbf{x},t-\tau)T(\mathbf{y},\tau)]\,dS_{\mathbf{y}}\,d\tau,\quad \mathbf{x}\in V,\quad t>0$$
$$Q^*(\mathbf{y},\mathbf{x},t-\tau) = \partial T^*(\mathbf{y},\mathbf{x},t-\tau)/\partial n^+(\mathbf{y})$$

Furthermore we replace x and y in (2.6) and set therein

$$X_i' = \delta_{im}\delta(\mathbf{y}-\mathbf{x}),\ G' = 0,\ u_i' = U_{im}(\mathbf{y},\mathbf{x})$$
$$p_i' = T_{im}(\mathbf{y},\mathbf{x}),\ T' = K_m^*(\mathbf{y},\mathbf{x},t-\tau),\ Q = N_m^*(\mathbf{y},\mathbf{x},t-\tau)$$

We here obtain an integral formula to represent the displacements for quasistatic UGT problems

$$u_m(\mathbf{x},t) = \int U_{im}(\mathbf{y},\mathbf{x})X_i(\mathbf{y},t)\,dV_{\mathbf{y}} + \int_0^t\int K_m^*(\mathbf{y},\mathbf{x},t-\tau)G(\mathbf{y},\tau)\,dV_{\mathbf{y}}\,d\tau + \tag{3.3}$$
$$\int c_\varepsilon(\mathbf{y})\{T^\circ(\mathbf{y})K_m^*(\mathbf{y},\mathbf{x},t) + \tau_t[T^1(\mathbf{y})K_m^*(\mathbf{y},\mathbf{x},t) +$$
$$T^\circ(\mathbf{y})\dot{K}_m^*(\mathbf{y},\mathbf{x},t)]\}\,dV_{\mathbf{y}} + \int_0^t\int [K_m^*(\mathbf{y},\mathbf{x},t-\tau)Q(\mathbf{y},\tau) -$$
$$N_m^*(\mathbf{y},\mathbf{x},t-\tau)T(\mathbf{y},\tau)]\,dS_{\mathbf{y}}\,d\tau + \int [U_{im}(\mathbf{y},x)p_i(\mathbf{y},t) -$$
$$T_{im}(\mathbf{y},\mathbf{x})u_i(\mathbf{y},t)]\,dS_{\mathbf{y}},\quad \mathbf{x}\in V,\quad t>0$$
$$N_m^*(\mathbf{y},\mathbf{x},t) = \partial K_m^*(\mathbf{y},\mathbf{x},t)/\partial n^+(\mathbf{y})$$

Formula (3.2) remains true even in the case of quasistatic problems.

The displacements and temperature within the body (i.e., for $\mathbf{x}\in V$) in terms of their boundary values and the boundary values of the stresses and heat flux are calculated by using the formulas obtained.

We note that since the first three equations in system (2.1) form a system of Lamé equations of motion (the system of equations of elastostatics in displacements in the case of quasistatics) for the isothermal case, the components $U_{im}(\mathbf{x},\mathbf{y},t)$, $(U_{im}(\mathbf{x},\mathbf{y}))$ of the fundamental solution of the adjoint system are also components of the fundamental solution of non-stationary isothermal elastodynamics (elastostatics).

Moreover, the components $T_{im}(\mathbf{x},\mathbf{y},t)$ $(T_{im}(\mathbf{x},\mathbf{y}))$ are identical with components of the same kind of isothermal elastodynamics (elastostatics).

We will now set up a relation between the components K_m, T^* (K_m^*, T^*) of the fundamental solution of system (2.1) (in the quasistatics case $\rho \ddot{u}_i \equiv 0$) and the fundamental solution of the initial system corresponding to a unit pulse heat source).

Assuming the body to be without limit and considering the initial conditions and mass forces to be zero and $G(\mathbf{y},\tau) = \delta(\mathbf{y}-\mathbf{x})\,\delta(\tau)$ we obtain from (3.1) and (3.3)

$$T_1^*(\mathbf{x},\mathbf{y},t) = T^*(\mathbf{y},\mathbf{x},t),\quad U_m^*(\mathbf{x},\mathbf{y},t) = K_m(\mathbf{y},\mathbf{x},t) \tag{3.4}$$
$$U_m''(\mathbf{x},\mathbf{y},t) = K_m^*(\mathbf{y},\mathbf{x},t)$$

Here $T_1^*(\mathbf{x}, \mathbf{y}, t)$, $U_m^*(\mathbf{x}, \mathbf{y}, t)$ and $U_m''(\mathbf{x}, \mathbf{y}, t)$ are components of the fundamental solutions of the initial systems, respectively.

4. Representation formulas for the stresses and heat flux. Replacing the notation of the subscript by l in (3.1), acting on both sides of this equality with the operator $C_{mrlk}\partial/\partial x_k$ and taking account of (1.2), we obtain a representation formula for the stresses

$$\sigma_{mr}(\mathbf{x}, t) = \int_0^t\int D_{mri}(\mathbf{x}, \mathbf{y}, t-\tau) X_i(\mathbf{y}, \tau)\, dV_\mathbf{y}\, d\tau + \tag{4.1}$$
$$\int_0^t\int V_{mr}^*(\mathbf{x}, \mathbf{y}, t-\tau) G(\mathbf{y}, \tau)\, dV_\mathbf{y}\, d\tau + \int\{\rho(\mathbf{y})[\dot{D}_{mri}(\mathbf{x}, \mathbf{y}, t) u_i^\circ(\mathbf{y}) +$$
$$D_{mri}(\mathbf{x}, \mathbf{y}, t) u_i^1(\mathbf{y})] + c_\varepsilon(\mathbf{y})[T^\circ(\mathbf{y}) V_{mr}^*(\mathbf{x}, \mathbf{y}, t) + \tau_t T^1(\mathbf{y}) V_{mr}^*(\mathbf{x}, \mathbf{y}, t) +$$
$$\tau_t T^\circ(\mathbf{y}) \dot{V}_{mr}(\mathbf{x}, \mathbf{y}, t)]\}\, dV_\mathbf{y} + \int_0^t\int [V_{mr}^*(\mathbf{x}, \mathbf{y}, t-\tau) Q(\mathbf{y}, \tau) -$$
$$S_{mr}^*(\mathbf{x}, \mathbf{y}, t-\tau) T(\mathbf{y}, \tau)]\, dS_\mathbf{y}\, d\tau + \int_0^t\int [D_{mri}(\mathbf{x}, \mathbf{y}, t-\tau) p_i(\mathbf{y}, \tau) -$$
$$S_{mri}(\mathbf{x}, \mathbf{y}, t-\tau) u_i(\mathbf{y}, \tau)]\, dS_\mathbf{y}\, d\tau - \beta_{mr} T(\mathbf{x}, t), \quad \mathbf{x} \in V, \quad t > 0$$
$$D_{mri}(\mathbf{x}, \mathbf{y}, t) = C_{mrlk}(\mathbf{x})\, \partial U_{li}(\mathbf{x}, \mathbf{y}, t)/\partial x_k$$
$$S_{mri}(\mathbf{x}, \mathbf{y}, t) = C_{mrlk}(\mathbf{x})\, \partial T_{li}(\mathbf{x}, \mathbf{y}, t)/\partial x_k$$
$$V_{mr}^*(\mathbf{x}, \mathbf{y}, t) = C_{mrlk}(\mathbf{x})\, \partial U_l^*(\mathbf{x}, \mathbf{y}, t)/\partial x_k$$
$$S_{mr}^*(\mathbf{x}, \mathbf{y}, t) = C_{mrlk}(\mathbf{x})\, \partial N_l^*(\mathbf{x}, \mathbf{y}, t)/\partial x_k$$

where

$$T_{li}(\mathbf{x}, \mathbf{y}, t) = C_{ijkm} n_j(\mathbf{y})\, \partial U_{ml}(\mathbf{y}, \mathbf{x}, t)/\partial y_k$$

Acting on both sides of the equality (3.2) with the operator $\partial/\partial n^+(\mathbf{x})$ we obtain in representation formula for the heat flux

$$Q(\mathbf{x}, t) = \int_0^t\int \frac{\partial}{\partial n^+(\mathbf{x})} T_1^*(\mathbf{x}, \mathbf{y}, t-\tau) G(\mathbf{y}, \tau)\, dV_\mathbf{y}\, d\tau + \tag{4.2}$$
$$\int c_\varepsilon(\mathbf{y}) \frac{\partial}{\partial n^+(\mathbf{x})} T_1^*(\mathbf{x}, \mathbf{y}, t) T^\circ(\mathbf{y})\, dV_\mathbf{y} + \int \tau_t c_\varepsilon(\mathbf{y}) \frac{\partial}{\partial n^+(\mathbf{x})} T_1^*(\mathbf{x}, \mathbf{y}, t) T^1(\mathbf{y})\, dV_\mathbf{y} +$$
$$\int \tau_t c_\varepsilon(\mathbf{y}) \frac{\partial}{\partial n^+(\mathbf{x})} \dot{T}_1^*(\mathbf{x}, \mathbf{y}, t) T^\circ(\mathbf{y})\, dV_\mathbf{y} + \int_0^t\int \left[\frac{\partial}{\partial n^+(\mathbf{x})} T_1^*(\mathbf{x}, \mathbf{y}, t-\tau) Q(\mathbf{y}, \tau) -\right.$$
$$\left.\frac{\partial}{\partial n^+(\mathbf{x})} Q_1^*(\mathbf{x}, \mathbf{y}, t-\tau) T(\mathbf{y}, \tau)\right] dS_\mathbf{y}\, d\tau, \quad x \in V, \quad t > 0$$
$$Q_1^*(\mathbf{x}, \mathbf{y}, t-\tau) = \partial T_1^*(\mathbf{x}, \mathbf{y}, t-\tau)/\partial n^+(\mathbf{x})$$

In the case of quasistatic problems the formula for the stresses has the form

$$\sigma_{mr}(\mathbf{x}, t) = \int D_{mri}(\mathbf{x}, \mathbf{y}) X_i(\mathbf{y})\, dV_\mathbf{y} + \int_0^t\int V_{mr}^*(\mathbf{x}, \mathbf{y}, t-\tau) G(\mathbf{y}, \tau)\, dV_\mathbf{y}\, d\tau + \tag{4.3}$$
$$\int c_\varepsilon(\mathbf{y}) V_{mr}^*(\mathbf{x}, \mathbf{y}, t) T^\circ(\mathbf{y})\, dV_\mathbf{y} + \int \tau_t c_\varepsilon(\mathbf{y}) V_{mr}^*(\mathbf{x}, \mathbf{y}, t) T^1(\mathbf{y})\, dV_\mathbf{y} +$$
$$\int \tau_t c_\varepsilon(\mathbf{y}) \dot{V}_{mr}^*(\mathbf{x}, \mathbf{y}, t) T^\circ(\mathbf{y})\, dV_\mathbf{y} + \int_0^t\int [V_{mr}^*(\mathbf{x}, \mathbf{y}, t-\tau) Q(\mathbf{y}, \tau) -$$
$$S_{mr}^*(\mathbf{x}, \mathbf{y}, t-\tau) T(\mathbf{y}, \tau)]\, dS_\mathbf{y}\, d\tau + \int [D_{mri}(\mathbf{x}, \mathbf{y}) p_i(\mathbf{y}, t) -$$
$$S_{mri}(\mathbf{x}, \mathbf{y}) u_i(\mathbf{y}, t)]\, dS_\mathbf{y} - T(\mathbf{x}, t)\beta_{mr}, \quad \mathbf{x} \in V, \quad t > 0$$
$$D_{mri}(\mathbf{x}, \mathbf{y}) = C_{mrlk}(\mathbf{x})\, \partial U_{li}(\mathbf{x}, \mathbf{y})/\partial x_k$$
$$S_{mri}(\mathbf{x}, \mathbf{y}) = C_{mrlk}(\mathbf{x})\, \partial T_{li}(\mathbf{x}, \mathbf{y})/\partial x_k$$
$$V_{mr}^*(\mathbf{x}, \mathbf{y}, t) = C_{mrlk}(\mathbf{x})\, \partial U_l''(\mathbf{x}, \mathbf{y}, t)/\partial x_k$$
$$S_{mr}^*(\mathbf{x}, \mathbf{y}, t) = C_{mrlk}(\mathbf{x})\, \partial N_l''(\mathbf{x}, \mathbf{y}, t)/\partial x_k$$

where

$$T_{li}(\mathbf{x}, \mathbf{y}) = C_{ijkm} n_j(\mathbf{y})\, \partial U_{ml}(\mathbf{y}, \mathbf{x})/\partial y_k$$

The stresses and heat flux at interior points of the domain under consideration can be calculated for appropriate boundary-value problems by using the representations (4.1)-(4.3).

Boundary-time integral equations for the above-mentioned initial-boundary value problems can be constructed in a direct formulation on the basis of the integral representations given above (when there are fundamental and singular solutions). Moreover, the approach elucidated can be extended to two-dimensional initial-boundary value UGT problems on replacing the fundamental solutions.

The author is grateful to R.V. Gol'dshtein for his interest.

REFERENCES

1. NOWACKI W., Dynamic Thermoelasticity Problems, Mir, Moscow, 1970.
2. BURCHULADZE T.V. and GEGELIA T.G., Development of the Potential Method in Elasticity Theory. Metsniereba, Tbilisi, 1985.
3. KUKUDZHANOV V.N. and OSTRIK A.V., Dynamical problems of coupled thermoelasticity, Plasticity and Fracture of Solids, Nauka, Moscow, 1988.
4. IGNACZAK J., On a three-dimensional solution of dynamic thermoelasticity with two relaxation times, J. Therm. Stresses, 4, 3-4, 1981.
5. KHUTORYANSKII N.M., On potential theory for non-stationary dynamic problems of uncoupled thermoelasticity, Applied Strength and Plasticity Problems, 15, Izd. Gor'k. Univ., Gor'kii, 1980.

Translated by M.D.F.

PMM U.S.S.R., Vol.54, No.6, pp. 820-824, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00
©1992 Pergamon Press plc

A PROBLEM IN ELASTICITY THEORY*

V.A. YURKO

The problem of determining the dimensions of the transverse cross-sections of a beam from the given frequencies of its natural vibrations is examined. Frequency spectra are indicated that determine the dimenions of the transverse cross-sections of the beam uniquely, an effective procedure is presented for solving the inverse problem, and a uniqueness theorem is proved. The method of standard models /1/ is used to solve the inverse problem.

We examine the differential equation describing beam vibrations in the form

$$(h^{\mu}(x)\,y'')'' = \lambda h(x)\,y, \quad 0 \leqslant x \leqslant T \tag{1}$$

here $h(x)$ is a function characterizing the beam transverse section, and $\mu = 1, 2, 3$ is a fixed number. We will assume that the function $h(x)$ is absolutely continuous in the segment $[0, T]$ and $h(x) > 0$, $h(0) = 1$. The inverse problem for (1) in the case $\mu = 2$ (similar transverse sections) was investigated /2/ in determining small changes in the beam transverse sections for given small changes in a finite number of its natural vibration frequencies.

Let $\{\lambda_{kj}\}_{k\geqslant 1, j=1,2}$ be the eigenvalues of boundary-value problems Q_j for (1) with the boundary conditions

$$y(0) = y^{(j)}(0) = y(T) = y'(T) = 0$$

The inverse problem is formulated as follows.

Problem 1. Find the function $h(x)$, $x \in [0, T]$ for given frequency spectra $\{\lambda_{kj}\}_{k\geqslant 1, j=1,2}$.

To solve this inverse problem we will first prove several auxiliary assertions.

We consider the function $\Phi(x, \lambda)$ the solution of (1) under the conditions $\Phi(0, \lambda) = \Phi(T, \lambda) = \Phi'(T, \lambda) = 0$, $\Phi'(0, \lambda) = 1$. We set $\alpha(\lambda) = \Phi''(0, \lambda)$. Furthermore, let the functions $C_\nu(x, \lambda)$ $(\nu = 0, 1, 2, 3)$ be solutions of (1) under the initial conditions $C_\nu^{(\mu)}(0, \lambda) = \delta_{\nu\mu}$, $\nu, \mu = 0, 1, 2, 3$. We will use the notation $\Delta_j(\lambda) = C_{3-j}(T, \lambda)\, C_3'(T, \lambda) - C_3(T, \lambda)\, C_{3-j}'(T, \lambda)$, $j = 1, 2$

Prikl. Matem. Mekhan., 54, 6, 998-1002, 1990